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Differential Quadrature Analysis of Stability and Free Vibration for Tapered Bernoulli Beam on Two Parameter Elastic Foundation

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Abstract: *Differential quadrature method (DQM) is used to analyze the stability and the free vibration behavior of axially loaded simply supported tapered Bernoulli beam rested on a two parameter elastic foundation. The depth of the beam is assumed to increase linearly from one end till the midpoint of the beam and then decrease linearly to the other end, whereas the width of the beam is assumed to be constant. A discretization has been done to the governing differential equation at sampling points, then homogeneous algebraic equations are obtained by substituting the implemented boundary conditions into the governing differential equation. After that, equivalent two-parameter eigenvalue problem is obtained and solved for the stability parameter for static case and the frequency parameter for dynamic case. The obtained solutions are verified against those from other technique. The effects of different parameters on the critical loads and the frequencies parameters are investigated.*

Keywords: *Tapered Bernoulli beam, simply supported, differential quadrature, axial load, and natural frequencies.*

1. Introduction

Beams are classified according to their profile (the shape of their cross-section), their length, and their material. The weight of the structural elements have a great influence on justify the functional requirements of the structure. Therefore, Tapered elements are generally used in many practical applications to optimize the weight of the structural elements. The static and dynamic behavior of such elements need design criteria to identify the optimal configurations. The analytical solution of tapered structural elements is very hard to obtain due to the complication of the partial differential governing equations. Therefore numerical methods are used to solve these complicated equations. Different configurations for simple cases are investigated by several researchers to acquire stability and/or vibration behaviors of such structural elements [1-3]. Taha et.al [4] solved the free vibration of non-uniform beam resting on elastic foundation using Bessel functions. Ruta [5] used the Chebychev series to study the vibration of non-prismatic beam. Sato [6] studied the effect of end restraints and axial force on the vibration frequencies for tapered beams using Ritz method. Rosa et.al [7] introduced dynamic stability of Euler-Bernoulli beam on two parameter elastic soil subjected to partially tangential forces in the presence of elastically flexible constraints. Seong-Min Kim [8] introduced the vibration analysis of an infinite Bernoulli-Euler beam resting on Winkler type elastic foundation. Bichir and Nassar [9] solved the problem of the free and forced vibrations of a delaminated beam-plate resting on elastic foundation.

Another numerical methods such as finite element method [11-12] and differential quadrature method (DQM) [12-17] are used to study certain configurations of such models.
In the present paper, the stability and vibration behavior of axially loaded simply supported tapered beams are studied using the DQM. The obtained solutions were verified against the FEM solution [4] and found in close agreement. The main differences between the present work and the previous one are the method of solution. The effects of different parameters related to the studied model on the load and frequency parameters are investigated.

2. Problem Formulation:

2.1. Vibration Equation:

Consider a linearly tapered beam with variable depth d(x), constant width (b) and length (L) is resting on a two-parameter elastic foundation and subjected to an axial force (Po) and vertical dynamic load q(x,t) with pinned-pinned supports as shown in Fig.(1) and Fig.(2).

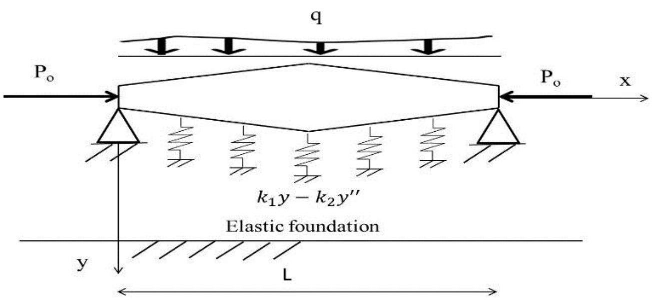

Fig. 1 Axially Loaded Pinned-Pinned tapered beam resting on two parameter foundation

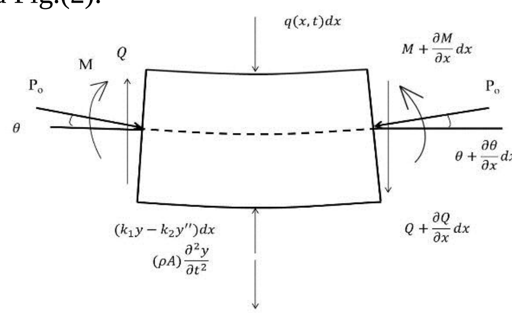

Fig. 2: Forces acting on differential element of the beam

Where k_1 is the linear coefficient of elastic foundation (linear stiffness); k_2 is the nonlinear coefficient of elastic foundation (nonlinear stiffness); Q (x) is the vertical shear force; M (x) is internal moment; ρ is density of the beam per unit volume; A(x) is area of the beam cross section at distance x; I(x) is moment of inertia of the beam cross section at distance x; E is the young's modulus of the beam material; y(x, t) is the lateral displacement of the beam, x is the distance along the beam; and t is the time.
The partial differential equation (PDE) for Bernoulli beam is given as:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + (P_o - k_2)\frac{\partial^2 y}{\partial x^2} + k_1 y + \rho A\frac{\partial^2 y}{\partial t^2} = q(x,t) \tag{1}$$

For free vibration $q(x,t) = 0$, then eqn. (1) leads to:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + (P_o - k_2)\frac{\partial^2 y}{\partial x^2} + k_1 y + \rho A\frac{\partial^2 y}{\partial t^2} = 0 \tag{2}$$

Convert eqn. (2) to dimensionless version by using dimensionless parameters X=*x*/L & W=*y*/L, then:

$$\frac{\partial^2}{\partial X^2}\left(\frac{EI}{L^3}\frac{\partial^2 W}{\partial X^2}\right) + \left(\frac{P_o - k_2}{L}\right)\frac{\partial^2 W}{\partial X^2} + k_1 LW + \rho AL\frac{\partial^2 W}{\partial t^2} = 0 \tag{3}$$

The lateral displacement W(X) is distributed into two independent functions, one for spatial variation (mode shape function $\Phi(X)$) and the other for time variation $\Psi(t)$ by using the separation of variable method. Then eqn. (3) became as shown in eqn. (4) and (5), where ω = separation constant.

$$\frac{1}{\rho AL^4\Phi(X)}\frac{d^2}{dX^2}\left(EI\frac{d^2\Phi}{dX^2}\right) + \frac{1}{\rho AL^2\Phi(X)}(P_o - k_2)\frac{d^2\Phi}{dX^2} + \frac{k_1}{\rho A} = \omega^2 \tag{4}$$

$$\ddot{\Psi} + \omega^2\Psi(t) = 0 \tag{5}$$

For initial conditions $\Psi(0) = 1$ and $\dot{\Psi}(0) = 0$, The solution of eqn.(5) equals $\Psi(t) = cos\omega t$.
For the tapered beam shown in Fig. (1), where the tapering ratio α is defined as $\alpha = d_I/d_o$,then the area and the moment of inertia at distance x are given as:

$$A(X) = \begin{cases} A_o[1 + (\alpha - 1) * 2X]\,; & 0 \le X \le 1/2 \\ A_o[1 + (\alpha - 1) * 2 * (1 - X)]; & 1/2 \le X \le 1 \end{cases} \tag{6}$$

$$I(X) = \begin{cases} I_o[1 + (\alpha - 1) * 2X]^3; & 0 \le X \le {}^1\!/_2 \\ I_o[1 + (\alpha - 1) * 2 * (1 - X)]^3; & {}^1\!/_2 \le X \le 1 \end{cases} \tag{7}$$

Substitution eqn. (6) and eqn. (7) into eqn. (4) gets the governing equation:

$$\frac{d^4\Phi}{dX^4} + (\beta)\frac{d^3\Phi}{dX^3} + (\eta_1 + \eta_2 P_o)\frac{d^2\Phi}{dX^2} + (\xi_1 - \xi_2\omega^2)\Phi(x) = 0 \tag{8}$$

$$\text{Where } \beta = \frac{2}{I(X)}\frac{dI(X)}{dX}\ ,\ \ \eta_1 = \frac{1}{I(X)}\frac{d^2I(X)}{dX^2} - \frac{k_2L^2}{EI(X)}\ ,\ \eta_2 = \frac{L^2}{EI(X)}\ ,\ \ \xi_1 = \frac{k_1L^2}{EI(X)}\ ,\ \ \xi_2 = \frac{\rho AL^4}{EI(X)} \tag{9}$$

Then the PDE is transformed into a homogeneous system of N algebraic equations by using DQM. However, the solution of the linear version of eqn. (3) depends on the boundary conditions at the beam ends.

2.2. Boundary Conditions:

The simply supported end condition (Pinned-Pinned (P-P)) in dimensionless form are expressed as:

$$\Phi = \frac{\mathrm{d}^2\Phi}{\mathrm{dX}^2} = 0, \qquad \text{at} \quad \mathrm{X} = 0 \tag{10}$$

$$\Phi = \frac{\mathrm{d}^2\Phi}{\mathrm{dX}^2} = 0, \qquad \text{at} \quad \mathrm{X} = 1 \tag{11}$$

3. Problem Solution Using DQM:

3.1.Differential Quadrature Method (DQM):

The solution of eqn. (8) is obtained using the DQM, where the solution domain is discretized into N sampling points and the derivatives at any point are approximated by a weighted linear summation of all the functional values at the other points [12].

$$\left.\frac{d^m f(x)}{d^m x}\right|_{x_i} \approx \sum_{j=1}^{N} C_{i,j}^{m}\ .\ f(x_j), \qquad for\ i = 1,2,\dots,N\ and\ m = 1,2,\dots,M \tag{12}$$

Where M is the order of the highest derivative in the governing equation, $f(x_j)$ is the functional value at point of x=x_j and $C_{i,j}^{m}$ are the weighting coefficients relating the derivative m at x=x_i to the functional value at x=x_j. To get the weighting coefficients, many polynomials with different base functions can be used. Lagrange interpolation formula is used, where the functional value at a point x is approximated by all the functional values $f(x_k)$, (k=1, N) as:

$$f(x) = \sum_{k=1}^{N} \frac{\prod_{j=1}^{N}(x - x_j)}{(x - x_k)\ .\ \prod_{k=1}^{N}(x_i - x_k)} f(x_k) \qquad i = 1,2,\dots,N\ \&\ k = 1,2,\dots,N \tag{13}$$

Substitution of eqn. (18) into eqn.(17) gets the weighting coefficients of the first derivative as [12]:

$$C_{i,j}^{(1)} = \frac{\prod_{k=1}^{N}(x_i - x_k)}{(x_i - x_j)\prod_{k=1}^{N}(x_j - x_k)} \quad for\ (i \ne j)\ and\ (i,j = 1,N) \tag{14}$$

$$C_{i,j}^{(1)} = -\sum_{j=1, j\ne i}^{N} C_{i,j}^{1} \quad for\ (i = j)\ and\ (i,j = 1,N) \tag{15}$$

Applying the chain rule on eqn.(17), the weighting coefficients of the (m) order expresses as:

$$C_{i,k}^{(m)} = -\sum_{j=1, j\ne i}^{N} C_{i,k}^{(1)} C_{i,k}^{(m-1)} \qquad for\ (i,j = 1,N)\ and\ (m = 1,M) \tag{16}$$

As the DQM is a numerical method, its accuracy is affected by both the number and the distribution of discretization points. In boundary value problems, it is found that the irregular distribution of the discretization points with smaller mesh spaces near the boundary to cope the steep variation near the boundaries is more adaptable. One of the frequently used distributions for mesh points is the normalized Gauss-Chebychev – Lobatto distribution given as:

$$x_i = \frac{1}{2}\left[1 - cos\left(\frac{(i-1)\pi}{N-1}\right)\right]\ ,\ i = 1,2,\dots,N \tag{17}$$

3.2.Discretization of Boundary Condition:

The boundary condition at beam ends may be written in the DQM discretized form as:

$$\Phi_1 = 0 \tag{18}$$
$$\Phi_2 = \frac{1}{AXN_{PP}} \sum_{k=3}^{N-2} AXK1_{PP}\ .\Phi_k \tag{19}$$
$$\Phi_{N-1} = \frac{1}{AXN_{PP}} \sum_{k=3}^{N-2} AXKN_{PP}\ .\Phi_k \tag{20}$$
$$\Phi_N = 0 \tag{21}$$

Where
$$AXK1_{PP} = C_{1,k}^{(1)}\,.C_{N,N-1}^{(1)} - C_{1,N-1}^{(1)}\,.C_{N,k}^{(1)} \tag{22}$$
$$AXKN_{PP} = C_{1,2}^{(1)}\,.C_{N,k}^{(1)} - C_{1,k}^{(1)}\,.C_{N,2}^{(1)} \tag{23}$$
$$AXN_{PP} = C_{N,2}^{(1)}\,.C_{1,N-1}^{(1)} - C_{1,2}^{(1)}\,.C_{N,N-1}^{(1)} \tag{24}$$

Expressing the unknown functional values at beam ends,Φ_1; Φ_2; Φ_{N-1} and Φ_N in terms of the other functional values, ϕ_i, (i=3, N-2), then the governing equation can be discretized at N-4 points yielding a system of N-4 homogeneous algebraic equations in N-4 unknown function values , ϕ_i, (i=3, N-2), in addition to the parameters of axial load and vibration natural frequency.

3.3.Discretization of Governing Equation:

Using the DQM, the governing equation of can be discretized at N-4 sampling points as:

$$\sum_{k=1}^{N} C_{i,k}^{(4)}\Phi_k + \beta(X_i)\sum_{k=1}^{N} C_{i,k}^{(3)}\Phi_k + [\eta_1(X_i) + \eta_2(X_i)P_o]\sum_{k=1}^{N} C_{i,k}^{(2)}\Phi_k - [\xi_1(X_i) - \xi_2(X_i)\omega^2]\Phi_i = 0\,, \quad for\ \ i = 3,4,\dots,N-2 \tag{25}$$

Then, using Lagrange interpolation polynomial and then obtains:

$$\sum_{k=3}^{N-2}\left(C_{1_k} + \beta_k C_{2_k} + \eta_{1_k} + \eta_{2_k}P_o - \left(\xi_1 - \xi_{2_k}\omega^2\right)\delta_{ik}\right)C_{3_k}\Phi_k = 0 \tag{26}$$

Where
$$\delta_{ik} = kronecr\ delta \begin{cases} 1 & i = k \\ o & i \neq k \end{cases} \tag{27}$$

C_1, C_2, C_3 are parameters introduced to simplify the obtained equation

$$C_1 = C_{i,k}^{(4)} - \left[\frac{(AXK1_{pp})C_{i,2}^{(4)} + (AXKN_{pp})C_{i,N-1}^{(4)}}{(AXN_{pp})}\right] \tag{28}$$
$$C_2 = C_{i,k}^{(3)} - \left[\frac{(AXK1_{pp})C_{i,2}^{(3)} + (AXKN_{pp})C_{i,N-1}^{(3)}}{(AXN_{pp})}\right] \tag{29}$$
$$C_3 = C_{i,k}^{(2)} - \left[\frac{(AXK1_{pp})C_{i,2}^{(2)} + (AXKN_{pp})C_{i,N-1}^{(2)}}{(AXN_{pp})}\right] \tag{30}$$

Eqn. (25) represents a homogeneous system of N-4 equations with two parameters (P_o and ω). Assigning a value for one of the two parameters leads to Eigenvalue problem, which can be solved to obtain the value of the other parameter. However, to calculate the critical axial load P_{cr}, the ω is assumed zero to eliminate the inertia term in governing equation. For ω calculations, appropriate value for axial load ($P_o<P_{cr}$) is assumed. A MATLAB program has been designed to solve the non-dimensional system eqn. (25) and calculating critical loads, vibration natural frequencies and the functional values of dimensionless lateral displacement at different locations along the beam.

3.4. Verification of present solution:

Many comparisons for simple cases are introduced, each of them verify certain parameter of the present problem. Introducing the following dimensionless parameters for both linear and nonlinear coefficient of subgrade reaction:

$$\bar{k}_1 = \frac{k_1 L^4}{EI_o} \tag{31}$$
$$\bar{k}_2 = \frac{k_2 L^2}{\pi^2 EI_o} \tag{32}$$

Defining the loading ratio γ as: $\gamma = \frac{P_o}{P_{cr}}$ (33)

Where P_{cr} is the critical load for the actual beam.

3.4.1. Comparison with analytical results for prismatic beam:

To verify the present solution, values of the frequency parameter λ and stability parameter λ_b for the case of simply supported prismatic beam are calculated and compared with exact solutions [4] in Table (I). To obtain the critical load P_{cr}, the eigenvalue problem is solved assuming zero natural frequency (ω).

TABLE I: values of frequency parameter for prismatic beam

	Frequency Parameter $\left(\lambda = \sqrt[4]{\frac{\rho A_o \omega^2 L^4}{EI_o}}\right)$	Stability Parameter $\left(\lambda_b = \sqrt{\frac{P_{cr} L^2}{EI_o}}\right)$
Exact Solution	3.141	$\pi = 3.1415$
DQM	3.141	3.1413

3.4.2. Comparison with FEM results for tapered beam:

Values of the stability parameter λ_b calculated using the present analysis are compared with values obtained using the finite element method [4] for the case of prismatic beam with pinned -pinned supports resting on two-parameter elastic foundation. The results indicate close agreement between the two approaches as shown in Table (II).

TABLE II: Comparisons of values of stability parameter for prismatic beams

$\bar{k}_1$	$\bar{k}_2$							
	0		0.5		1		2.5	
	FEM	Present	FEM	Present	FEM	Present	FEM	Present
0	3.1415	3.1414	3.8475	3.8375	4.4428	4.4425	5.8774	5.8719
1	3.1576	3.1567	3.8609	3.8604	4.4543	4.4544	5.8860	5.8794
10^2	4.4723	4.4642	4.9936	4.9868	5.4654	5.4595	6.6840	6.6799
10^4	14.191	14.204	14.364	14.377	14.535	14.547	15.036	15.025

Another attempt to verify the present solution, values of the frequency parameter λ for Clamped – Clamped prismatic beam are calculated using the present analysis and compared with values obtained by using finite element method [4] and found in close agreement as shown in Table (III).

TABLE III: values of frequency parameter for tapered beam

$\bar{k}_1$	γ	$\bar{k}_2$							
		0		0.5		1		2.5	
		FEM	Present	FEM	Present	FEM	Present	FEM	Present
0	0.0	3.1415	3.1402	3.4767	3.4745	3.7306	3.7342	4.2970	4.2949
	0.2	2.9734	2.9707	3.2906	3.2891	3.5306	3.5313	4.0669	4.0640
	0.4	2.7705	2.7622	3.0661	3.0598	3.2947	3.2859	3.7893	3.7819
	0.6	2.5097	2.4870	2.7773	2.7694	2.9842	2.9707	3.4319	3.4195
	0.8	2.1257	2.1020	2.3520	2.3363	2.5270	2.4990	2.9050	2.8787
10^2	0.0	3.7483	3.7475	3.9608	3.9607	4.1437	4.1414	4.5824	4.5819
	0.2	3.5477	3.5454	3.7487	3.7449	3.9218	3.9190	4.3370	4.3319
	0.4	3.3055	3.3010	3.4928	3.487	3.6541	3.6479	4.0408	4.0318
	0.6	2.9940	2.9840	3.1635	3.1535	3.3095	3.2980	3.6594	3.6451
	0.8	2.5350	2.5129	2.6782	2.6555	2.8014	2.7765	3.0964	3.0683
10^4	0.0	10.024	10.024	10.036	10.036	10.048	10.048	10.084	10.083
	0.2	9.9249	9.9230	9.935	9.9331	9.9449	9.9431	9.9746	9.9731
	0.4	9.5938	9.5663	9.6274	9.5986	9.6608	9.6317	9.7585	9.7317
	0.6	9.0933	9.0748	9.1219	9.1011	9.1500	9.1262	9.2314	9.2077
	0.8	8.0156	7.7318	8.0600	7.7781	8.1034	7.8240	8.2276	7.9816

4. Numerical Results:

Values for the stability parameter are calculated for different values of tapering ratio $\alpha = d_1/d_o$ and presented in Table (IV). The stability parameter represents the stiffness of the studied beam configuration against buckling due to axial load. Indeed, the stability parameter increases as the overall stiffness of the beam-foundation system increases. The overall stiffness of the beam foundation system composed of the flexural rigidity of the beam and the stiffness of the foundation.

TABLE IV: Variation of stability parameter λ_b with α and boundary condition ($\bar{k}_1 = 0$)

	λ_b					
	$\propto$=1	$\propto$=1.1	$\propto$=1.2	$\propto$=1.3	$\propto$=1.4	$\propto$=1.5
$\bar{k}_2 = 0$	3.1413	3.4631	3.768	4.0369	4.292	4.5309
$\bar{k}_2 = 2.5$	5.8644	6.0495	6.229	6.3966	6.506	6.7062

Figure (5.5) to Figure (5.7) show the effects of foundation parameters on stability parameter for Pinned – Pinned beam (P-P) for chosen values of tapering ratio α. It is clear that the stability parameter λ_b increases as the foundation parameters increase and as the tapering ratio increases. The significant influence of the foundation parameter appears for value of $\bar{k}1$ >100 and for values of $\bar{k}_2$>0. However the mutual influence of tapering ratio and foundation parameters is more noticeable for small values of α. The effects of foundation parameters and tapering ratio on stability of the beam-foundation system are shown in Figure (5.8) for wide range of these parameters. It is obvious that the effect of tapering ratio is more noticeable for small values of foundation parameters.

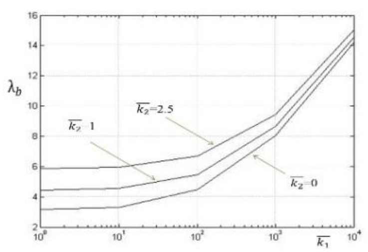

Fig. 3: Influence of the foundation parameters on the stability parameter in case of P-P beam for $\propto$=1

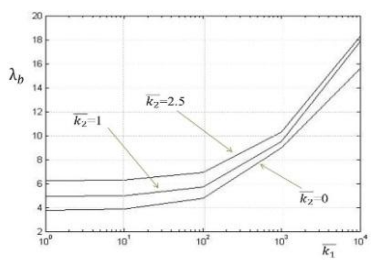

Fig. 4: Influence of the foundation parameters on the stability parameter in case of P-P beam for $\propto$=1.2

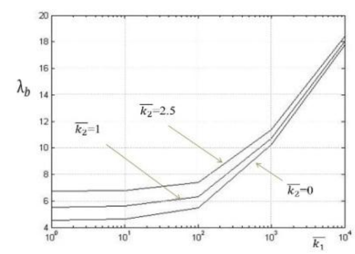

Fig. 5: Influence of the foundation parameters on the stability parameter in case of P-P beam for $\propto$=1.5

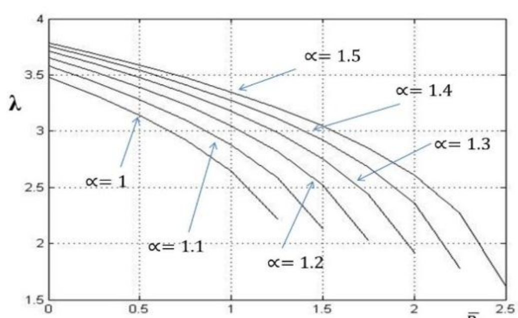

Fig. 6: Influence of the loading ratio on the frequency parameter for case of (P-P) ;($\bar{k}_1 = 0, \bar{k}_2 = 0.5$)

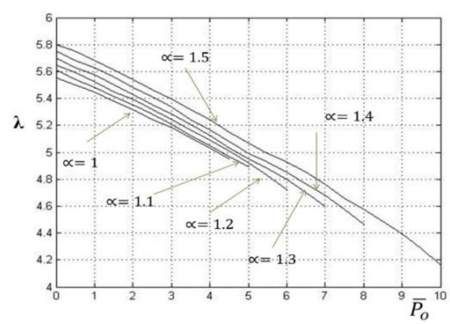

Fig. 7 Influence of the loading ratio on the frequency parameter for the case of (P-P);($\bar{k}_1 = 10^3, \bar{k}_2 = 0.5$)

The effect of loading ratio on the frequency parameter for the case of (P-P) case is shown in Fig. (5.20) to Fig. (5.21) for chosen values of the foundation parameters and the tapering ratio. It is clear that the frequency parameter increases as the loading ratio $\bar{P}_o$ decreases and as the foundation and tapering parameters increase. The effect of loading ratio is nearly linear for stiff foundation for all chosen values of the tapering ratio.

5. Conclusions

The stability and frequency parameters of axially-loaded simply supported tapered Bernoulli beam resting on a two-parameter elastic foundation were investigated using DQM. The obtained results found in a close agreement with both analytical method and FEM. It can be noted that increasing of both the natural frequencies and the critical loads led to increasing the integrated stiffness of the system. The integrated stiffness of the system represents a combination of the flexural stiffness of the beam and the stiffness of the foundation. The results also indicated that the natural frequency of the beam-foundation system increased when the applied axial load decreased, because the component of the compression axial load acted in the opposite direction of the restoring force produced from the overall stiffness of the beam foundation system.

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