

Free Vibration and Stability Behavior of Non-Prismatic Clamped-Clamped Beams Resting on Elastic Foundations Using DQM

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Abstract: *The stability behavior and the free vibration of axially loaded non-prismatic beams with clamped end restraints resting on elastic foundation were investigated using differential quadrature method (DQM). The governing differential equation was derived and discretized over the considered domain at N finite sampling points yielding system of N algebraic equations in N unknowns. Then, the eigenvalues was obtained by substituting the discretized boundary conditions to the governing differential equation, after that critical loads in the static case and natural frequencies in the dynamic case were calculated. The obtained solutions were verified against both analytical and FEM solution and found in close agreement. The effects of different parameters related to the studied model on the load and frequency parameters were investigated.*

Keywords: *non-prismatic beams, clamped end, differential quadrature, axial load, and natural frequencies.*

1. Introduction

Optimization the weight of structures elements is required for structural, architectural, and economic reasons. Therefore, non-prismatic structural elements are conducted to justify the functional requirement of the structures. It is difficult to get a closed form solution for non-uniform structural elements under static and dynamic loads. The analytical methods of such elements are hard to deal with due to the complicated governing equations while the numerical techniques offer dissolvable alternatives. Different configurations are studied by many researchers to obtain stability and/or vibration behaviors of such structural elements. Closed forms and analytical solutions obtained stability and/or vibration behaviors of prismatic and non-prismatic elements are found in literature [1-3].

Taha M.H. and Abohadima, S. [4] studied the free vibration of non-uniform shear beam resting on elastic foundation using Bessel functions. Ruta [5] used the Chebychev series to study the vibration of non-prismatic beam. Sato [6] studied the effect of end restraints and axial force on the vibration frequencies for tapered beams using Ritz method. Naidu et.al [7] presented the vibration of nonuniform beams resting on elastic foundation and obtained the frequencies and mode shapes using finite element method. Another numerical methods such as finite element method [8-9] and differential quadrature method (DQM) [10-14] are used to study certain configurations of such models.

In the present paper, the stability and vibration behavior of axially loaded non-prismatic beams with clamped end restraints were studied using the DQM. Beam depth was assumed to increase linearly from one end till the midpoint of the beam and then to decrease linearly to the other end, whereas the width of

the beam was considered constant. The obtained solutions were verified against the FEM solution [4] and found in close agreement. The main difference between the present work and the previous one was the method of solution. The effects of different parameters related to the studied model on the load and frequency parameters were investigated.

2. Problem Formulation:

2.1. Vibration Equation:

Consider a linearly tapered beam with variable depth $d(x)$, constant width (b) and length (L) is resting on a two-parameter elastic foundation and subjected to an axial force (P_o) with clamped-clamped supports as shown in Fig.(1) and Fig.(2).

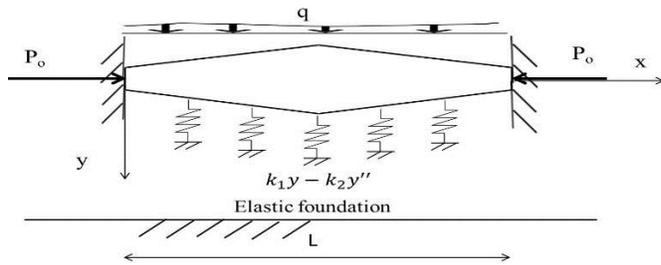


Fig. 1: Axially Loaded Clamped- Clamped tapered beam resting on two parameter foundation

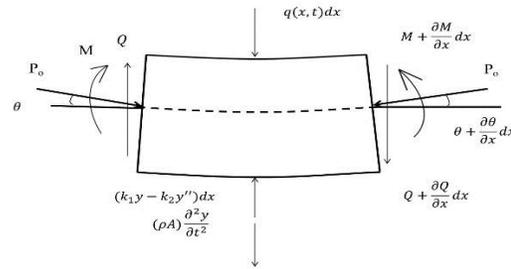


Fig. 2: Forces acting on differential element of the beam

Where k_1 is the linear coefficient of elastic foundation (linear stiffness); k_2 is the nonlinear coefficient of elastic foundation (nonlinear stiffness); $q(x, t)$ is the vertical dynamic load acting on the beam element; $Q(x)$ is the internal shear force; $M(x)$ is internal moment; ρ is density of the beam per unit volume; $A(x)$ is area of the beam cross section at distance x ; $I(x)$ is moment of inertia of the beam cross section at distance x ; $y(x, t)$ is the lateral displacement of the beam; and E is the young's modulus of the beam material.

The free vibration equation (at $q(x, t) = 0$) for the shown non-prismatic beam is given as:

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + (P_o - k_2) \frac{\partial^2 y}{\partial x^2} + k_1 y + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

Using dimensionless parameters $X=x/L$ & $W=y/L$, the partial differential equation (PDE) becomes:

$$\frac{\partial^2}{\partial X^2} \left(\frac{EI}{L^3} \frac{\partial^2 W}{\partial X^2} \right) + \left(\frac{P_o - k_2}{L} \right) \frac{\partial^2 W}{\partial X^2} + k_1 L W + \rho A L \frac{\partial^2 W}{\partial t^2} = 0 \quad (2)$$

For solving this linear PDE eqn. (2), first separation of variables is applied where the lateral displacement is distributed by two independent functions, one for spatial variation (mode shape function) and the other for time variation. Then DQM is applied to transform the governing PDE into a homogeneous system of N algebraic equations that are solved numerically. However, the solution of the linear version of eqn. (2) depends on the boundary conditions at the beam ends.

Using separation of variable method, the solution of the PDE (eqn. 2) may be assumed as:

$$W(X, t) = \Phi(X)\Psi(t) \quad (3)$$

Where $\Phi(X)$ is the linear mode function, $\Psi(t)$ is a function representing the variation with time. Substituting eqn. (3) into eqn. (2), eqn. (2) transform to eqn. (4), where $\omega =$ separation constant.

$$\frac{1}{\rho A L^4 \Phi(X)} \frac{d^2}{dX^2} \left(EI \frac{d^2 \Phi}{dX^2} \right) + \frac{1}{\rho A L^2 \Phi(X)} (P_o - k_2) \frac{d^2 \Phi}{dX^2} + \frac{k_1}{\rho A} = \omega^2 \quad (4)$$

$$\ddot{\Psi} + \omega^2 \Psi(t) = 0 \quad (5)$$

For initial conditions $\Psi(0) = 1$ and $\dot{\Psi}(0) = 0$, The solution of eqn.(5) equals $\Psi(t) = \cos\omega t$.

For the non-prismatic beam shown in Fig.(1) where the depth of the beam increases linearly from d_0 at $X=0$ to d_1 at $X=0.5$, then decreases linearly to d_0 at $X=1$, where the tapering ratio α is defined as $\alpha = d_1/d_0$, then the area and the moment of inertia at distance x are given as:

$$A(X) = \begin{cases} A_0[1 + (\alpha - 1) * 2X]; & 0 \leq X \leq 1/2 \\ A_0[1 + (\alpha - 1) * 2 * (1 - X)]; & 1/2 \leq X \leq 1 \end{cases} \quad (6)$$

$$I(X) = \begin{cases} I_0[1 + (\alpha - 1) * 2X]^3; & 0 \leq X \leq 1/2 \\ I_0[1 + (\alpha - 1) * 2 * (1 - X)]^3; & 1/2 \leq X \leq 1 \end{cases} \quad (7)$$

Substitution eqn. (6) and eqn. (7) into eqn. (4) gets the governing equation:

$$\frac{d^4\Phi}{dX^4} + (\beta) \frac{d^3\Phi}{dX^3} + (\eta_1 + \eta_2 P_0) \frac{d^2\Phi}{dX^2} + (\xi_1 - \xi_2 \omega^2) \Phi(x) = 0 \quad (8)$$

$$\text{Where } \beta = \frac{2}{I(X)} \frac{dI(X)}{dX}, \quad \eta_1 = \frac{1}{I(X)} \frac{d^2I(X)}{dX^2} - \frac{k_2 L^2}{EI(X)}, \quad \eta_2 = \frac{L^2}{EI(X)}, \quad \xi_1 = \frac{k_1 L^2}{EI(X)}, \quad \xi_2 = \frac{\rho A L^4}{EI(X)} \quad (9)$$

From the governing equation it's expected that the overall stiffness of the beam-foundation system increasing by increasing the beam flexural stiffness $(EI, E \frac{dI}{dX}, E \frac{d^2I}{dX^2})$ and by increasing the coefficients of foundation reaction (k_1, k_2) .

2.2. Boundary Conditions:

The dimensionless clamped-clamped end restraints at $X = 0$ are:

$$\Phi = \frac{d\Phi}{dX} = 0 \quad (10)$$

Similarly the dimensionless clamped end restraints at $X = 1$ are expressed as:

$$\Phi = \frac{d\Phi}{dX} = 0 \quad (11)$$

3. Problem Solution Using DQM:

3.1. Differential Quadrature Method (DQM):

The solution of eqn. (8) is obtained using the DQM, where the solution domain is discretized into N sampling points and the derivatives at any point are approximated by a weighted linear summation of all the functional values at the other points [12].

$$\left. \frac{d^m f(x)}{d^m x} \right|_{x_i} \approx \sum_{j=1}^N C_{i,j}^m \cdot f(x_j), \quad \text{for } i = 1, 2, \dots, N \text{ and } m = 1, 2, \dots, M \quad (12)$$

Where M is the order of the highest derivative in the governing equation, $f(x_j)$ is the functional value at point of $x=x_j$ and $C_{i,j}^m$ are the weighting coefficients relating the derivative m at $x=x_i$ to the functional value at $x=x_j$. To get the weighting coefficients, many polynomials with different base functions can be used. Lagrange interpolation formula is used, where the functional value at a point x is approximated by all the functional values $f(x_k)$, ($k=1, N$) as:

$$f(x) = \sum_{k=1}^N \frac{\prod_{j=1}^N (x-x_j)}{(x-x_k) \cdot \prod_{k=1}^N (x_i-x_k)} f(x_k) \quad i = 1, 2, \dots, N \ \& \ k = 1, 2, \dots, N \quad (13)$$

Substitution of eqn. (18) into eqn.(17) gets the weighting coefficients of the first derivative as [12]:

$$C_{i,j}^{(1)} = \frac{\prod_{k=1}^N (x_i-x_k)}{(x_i-x_j) \prod_{k=1}^N (x_j-x_k)} \quad \text{for } (i \neq j) \text{ and } (i, j = 1, N) \quad (14)$$

$$C_{i,j}^{(1)} = -\sum_{j=1, j \neq i}^N C_{i,j}^1 \quad \text{for } (i = j) \text{ and } (i, j = 1, N) \quad (15)$$

Applying the chain rule on eqn.(17), the weighting coefficients of the (m) order expresses as:

$$C_{i,k}^{(m)} = -\sum_{j=1, j \neq i}^N C_{i,k}^{(1)} C_{i,k}^{(m-1)} \quad \text{for } (i, j = 1, N) \text{ and } (m = 1, M) \quad (16)$$

As the DQM is a numerical method, its accuracy is affected by both the number and the distribution of discretization points. In boundary value problems, it is found that the irregular distribution of the discretization points with smaller mesh spaces near the boundary to cope the steep variation near the boundaries is more adaptable. One of the frequently used distributions for mesh points is the normalized Gauss-Chebyshev – Lobatto distribution given as:

$$x_i = \frac{1}{2} \left[1 - \cos \left(\frac{(i-1)\pi}{N-1} \right) \right], \quad i = 1, 2, \dots, N \quad (17)$$

3.2. Discretization of Boundary Condition:

The boundary condition at beam ends may be written in the DQM discretized form as:

$$\Phi_1 = 0 \quad (18)$$

$$\Phi_2 = \frac{1}{AXN_{cc}} \sum_{k=3}^{N-2} AXK1_{cc} \cdot \Phi_k \quad (19)$$

$$\Phi_{N-1} = \frac{1}{AXN_{cc}} \sum_{k=3}^{N-2} AXKN_{cc} \cdot \Phi_k \quad (20)$$

$$\Phi_N = 0 \quad (21)$$

Where

$$AXK1_{cc} = C_{1,k}^{(1)} \cdot C_{N,N-1}^{(1)} - C_{1,N-1}^{(1)} \cdot C_{N,k}^{(1)} \quad (22)$$

$$AXKN_{cc} = C_{1,2}^{(1)} \cdot C_{N,k}^{(1)} - C_{1,k}^{(1)} \cdot C_{N,2}^{(1)} \quad (23)$$

$$AXN_{cc} = C_{N,2}^{(1)} \cdot C_{1,N-1}^{(1)} - C_{1,2}^{(1)} \cdot C_{N,N-1}^{(1)} \quad (24)$$

Expressing the unknown functional values at beam ends, Φ_1 ; Φ_2 ; Φ_{N-1} and Φ_N in terms of the other functional values, Φ_i , ($i=3, N-2$), then the governing equation can be discretized at N-4 points yielding a system of N-4 homogeneous algebraic equations in N-4 unknown function values, Φ_i , ($i=3, N-2$), in addition to the parameters of axial load and vibration natural frequency.

3.3. Discretization of Governing Equation:

Using the DQM, the governing equation of can be discretized at N-4 sampling points as:

$$\sum_{k=1}^N C_{i,k}^{(4)} \Phi_k + \beta(X_i) \sum_{k=1}^N C_{i,k}^{(3)} \Phi_k + [\eta_1(X_i) + \eta_2(X_i)P_o] \sum_{k=1}^N C_{i,k}^{(2)} \Phi_k - [\xi_1(X_i) - \xi_2(X_i)\omega^2] \Phi_i = 0, \quad \text{for } i = 3, 4, \dots, N-2 \quad (25)$$

Then, using Lagrange interpolation polynomial and then obtains:

$$\sum_{k=3}^{N-2} \left(C_{1k} + \beta_k C_{2k} + \eta_{1k} + \eta_{2k} P_o - (\xi_1 - \xi_{2k} \omega^2) \delta_{ik} \right) C_{3k} \Phi_k = 0 \quad (26)$$

Where $\delta_{ik} = \text{kronocr delta} \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \quad (27)$

C_1, C_2, C_3 are parameters introduced to simplify the obtained equation

$$C_1 = C_{i,k}^{(4)} - \left[\frac{(AXK1_{cc})C_{i,2}^{(4)} + (AXKN_{cc})C_{i,N-1}^{(4)}}{(AXN_{cc})} \right] \quad (28)$$

$$C_2 = C_{i,k}^{(3)} - \left[\frac{(AXK1_{cc})C_{i,2}^{(3)} + (AXKN_{cc})C_{i,N-1}^{(3)}}{(AXN_{cc})} \right] \quad (29)$$

$$C_3 = C_{i,k}^{(2)} - \left[\frac{(AXK1_{cc})C_{i,2}^{(2)} + (AXKN_{cc})C_{i,N-1}^{(2)}}{(AXN_{cc})} \right] \quad (30)$$

Eqn. (25) represents a homogeneous system of N-4 equations with two parameters (P_o and ω). Assigning a value for one of the two parameters leads to Eigenvalue problem, which can be solved to obtain the value of the other parameter. However, to calculate the critical axial load P_{cr} , the ω is assumed zero to eliminate the inertia term in governing equation. For ω calculations, appropriate value for axial load ($P_o < P_{cr}$) is assumed. A MATLAB program has been designed to solve the non-dimensional system eqn. (25) and calculating critical loads, vibration natural frequencies and the functional values of dimensionless lateral displacement at different locations along the beam.

3.4. Verification of present solution:

Introducing the following dimensionless parameters for both linear and nonlinear coefficient of subgrade reaction:

$$\bar{k}_1 = \frac{k_1 L^4}{EI_o} \quad (31)$$

$$\bar{k}_2 = \frac{k_2 L^2}{\pi^2 EI_o} \quad (32)$$

Defining the loading ratio γ as:

$$\gamma = \frac{P_o}{P_{cr}} \quad (33)$$

Where P_{cr} is the critical load for the actual beam.

3.4.1. Comparison with Analytical Results for Prismatic Beam:

To verify the present solution, values of the frequency parameter λ for clamped end restraints of prismatic beam are calculated and compared with exact solutions [4] in Table (I).

TABLE I: values of frequency parameter for prismatic beam

	Frequency Parameter $\left(\lambda = \sqrt[4]{\frac{\rho A_o \omega^2 L^4}{EI_o}}\right)$	Stability Parameter $\left(\lambda_b = \sqrt{\frac{P_{cr} L^2}{EI_o}}\right)$
Exact Solution	4.730	$2\pi = 6.283$
DQM	4.717	6.2643

To obtain the critical load P_{cr} , the eigenvalue problem is solved assuming zero natural frequency (ω). The values of stability parameter λ_b for the case of prismatic beam with clamped end restraints also calculated from the DQM and found that it is close to the exact value as shown in Table (I).

3.4.2. Comparison with FEM Results for Tapered Beam:

TABLE II: Comparisons of values of stability parameter for prismatic beams

\bar{k}_1	\bar{k}_2							
	0		0.5		1		2.5	
	FEM	Present	FEM	Present	FEM	Present	FEM	Present
0	3.1415	3.1414	3.8475	3.8375	4.4428	4.4425	5.8774	5.8719
1	3.1576	3.1567	3.8609	3.8604	4.4543	4.4544	5.8860	5.8794
10^2	4.4723	4.4642	4.9936	4.9868	5.4654	5.4595	6.6840	6.6799
10^4	14.191	14.204	14.364	14.377	14.535	14.547	15.036	15.025
0	6.2832	6.2643	6.6643	6.6468	7.0248	7.0135	8.0095	7.9958
1	6.2892	6.2713	6.6701	6.6534	7.0302	7.0185	8.0142	8.0013
10^2	6.8561	6.8429	7.2070	7.1886	7.5416	7.5301	8.4664	8.4568
10^4	15.291	15.258	15.451	15.430	15.6105	15.601	16.077	16.047

TABLE III: values of frequency parameter for tapered beam

\bar{k}_1	γ	\bar{k}_2							
		0		0.5		1		2.5	
		FEM	Present	FEM	Present	FEM	Present	FEM	Present
0	0.0	4.7300	4.7186	4.8669	4.8545	4.9925	4.9801	5.3183	5.3060
	0.2	4.4739	4.4704	4.6039	4.6006	4.7231	4.7192	5.0324	5.0292
	0.4	4.1611	4.1630	4.2825	4.2810	4.3938	4.3958	4.6829	4.6917
	0.6	3.7508	3.7608	3.8608	3.8653	3.9618	3.9670	4.2240	4.2342
	0.8	3.1205	3.1560	3.2127	3.2493	3.2974	3.3400	3.5178	3.5706
10^2	0.0	4.9504	4.9403	5.0706	5.0594	5.1823	5.1704	5.4773	5.4660
	0.2	4.6843	4.6811	4.7985	4.7863	4.9046	4.9014	5.1847	5.1820
	0.4	4.3591	4.3502	4.4659	4.4632	4.5651	4.5713	4.8272	4.8277
	0.6	3.9323	3.9418	4.0292	4.0416	4.1193	4.1390	4.3574	4.3731
	0.8	3.2764	3.3100	3.3579	3.3991	3.4337	3.4859	3.6344	3.6805
10^4	0.0	10.122	10.119	10.137	10.136	10.151	10.150	10.194	10.192
	0.2	9.9729	9.9701	9.9860	9.9881	9.9990	10.000	10.036	10.039
	0.4	9.7932	9.7978	9.8066	9.809	9.8193	9.8191	9.8551	9.8596
	0.6	9.1176	9.1578	9.1507	9.1795	9.1833	9.2120	9.2784	9.3195
	0.8	8.1093	8.1569	8.1356	8.1691	8.1616	8.1934	8.2375	8.2901

Values of the stability parameter λ_b for the case of prismatic beam with clamped-clamped supports resting on two-parameter foundation are calculated using the present analysis. Then, these values are compared with values obtained from the finite element method [4] as shown in Table (II). The results indicate close agreement between the two approaches.

Another attempt to verify the present solution, values of the frequency parameter λ for Clamped – Clamped prismatic beam are calculated using the present analysis and compared with values obtained by using finite element method [4] and found in close agreement as shown in Table (III).

4. Numerical Results:

Values for the stability parameter are calculated for different values of tapering ratio $\alpha = d_1/d_0$ and presented in Table (IV). The stability parameter represents the stiffness of the studied beam configuration against buckling due to axial load. Indeed, the stability parameter increases as the overall stiffness of the beam-foundation system increases. The overall stiffness of the beam foundation system composed of the flexural rigidity of the beam and the stiffness of the foundation.

TABLE IV: Variation of stability parameter λ_b with α and boundary condition ($\bar{k}_1 = 0$)

	λ_b					
	$\alpha=1$	$\alpha=1.1$	$\alpha=1.2$	$\alpha=1.3$	$\alpha=1.4$	$\alpha=1.5$
$\bar{k}_2 = 0$	6.2643	7.2253	8.2509	9.3893	10.7069	12.5286
$\bar{k}_2 = 2.5$	7.9958	8.759	9.6225	10.5827	11.8038	13.4781

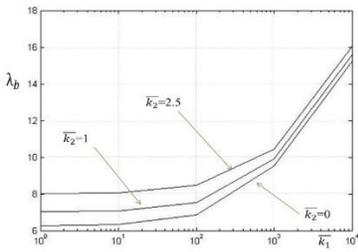


Fig. 3: Influence of the foundation parameters on the stability parameter in case of C-C beam for $\alpha = 1$

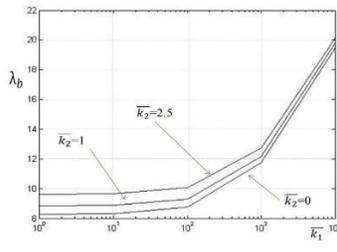


Fig. 4: Influence of the foundation parameters on the stability parameter in case of C-C beam for $\alpha = 1.2$

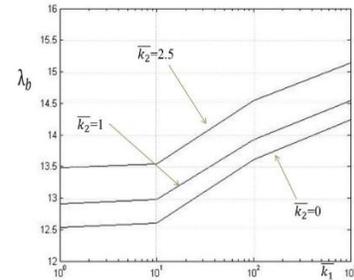


Fig. 5: Influence of the foundation parameters on the stability parameter in case of C-C beam for $\alpha = 1.5$

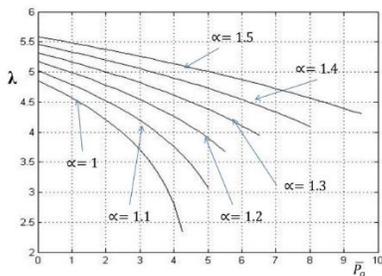


Fig. 6: Influence of the loading ratio on the frequency parameter for the case of (C-C) ($\bar{k}_1 = 0, \bar{k}_2 = 0.5$)

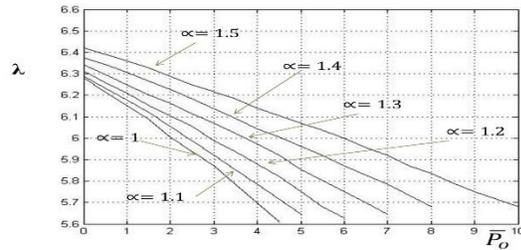


Fig. 7: Influence of the loading ratio on the frequency parameter for the case of (C-C); ($\bar{k}_1 = 10^3, \bar{k}_2 = 0.5$)

Fig. (3) to Fig. (5) show the effects of the foundation parameters on the stability parameter for Clamped – Clamped beam (C-C) for chosen values of tapering ratio α . The stability parameter λ_b increases as the foundation parameters increase and as the tapering ratio increases. The significant influence of the foundation parameter appears for value of $\bar{k}_1 > 100$ and for values of $\bar{k}_2 > 0$. Fig. (5.24) and (5.25) present the effects of loading ratio on the frequency parameter for chosen values of foundation parameters and tapering ratio. The frequency parameter increases as the loading ratio \bar{P}_0 decreases and as the foundation and tapering parameters increase. Also, the effect of the loading ratio approaches linear trend as the tapering ratio decreases and as the foundation parameters increase with steep variation for small values of the tapering ratio.

5. Conclusions:

The results of the DQM analysis showed that both stability parameter and frequency parameter increase as the overall stiffness of the system increases. The overall stiffness of the system composed of the flexural stiffness of the beam, which increases as α increases, and the stiffness of the foundation. It is also found that the effect of the tapering ratio α on the stability parameter is linear for stiff foundations, and it is has also linear trend on the frequency parameter for clamped-clamped condition. The frequency parameter λ increased as the applied axial load decreased, because the component of the compression axial load acted in the opposite direction of the restoring force produced from the overall stiffness of the beam foundation system.

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